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On some generalized American style derivatives

Tsvetelin S. Zaevski[1,2]

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Abstract
The aim of this paper is to examine some American style financial instruments with generalized payment structures, in particular for the power functions. We first prove several propositions related to the optimal regions and the corresponding early exercise boundaries. Based on them we derive the boundary values when the time to maturity is zero or infinitely large. We present also a numerical algorithm for approximating the whole boundary. Thus we can view the arising free boundary equation as a boundary value problem in a known region. We apply to it the Crank–Nicolson finite difference approach to derive the fair prices. Some numerical experiments are provided.

Keywords American derivatives · Optimal boundaries · Dividend rates · Perpetual derivatives · Finite maturities · Power payment functions

Mathematics Subject Classification 35R35 · 35Q91 · 60G44 · 91G20

1 Introduction

The American style financial derivatives are one of the most traded instruments at the modern markets. Their main feature is the existing prematurely exercise right. This way the holder has the opportunity to maximize the profit choosing the moment at which to activate the contract. Thus the evaluation of such derivatives turns to an optimal stopping problem. On the other hand, it can be viewed as a free boundary differential problem—we refer to Bather (1970), Moerbeke (1973), Wong (1996), Peskir and Shiryaev (2006), and Shiryaev (2009). Some of the most popular among these instruments are the American options. Several classical works consider American options in this framework—see Kim (1990), Jacka (1991), Carr

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✉ Tsvetelin S. Zaevski
t_s_zaevski@math.bas.bg; t_s_zaevski@abv.bg

1 Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Acad. Georgi Bonchev Str., Block 8, 1113 Sofia, Bulgaria

2 Faculty of Mathematics and Informatics, Sofia University “St. Kliment Ohridski”, boul. James Boucher 5, 1164 Sofia, Bulgaria

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et al. (1992), and Pham (1997). In addition many new works appear recently—we refer to Chernogorova et al. (2018), Abdi-Mazraeh and Khani (2018), Moradipour and Yousefi (2018), Heidari and Azari (2018), Zhang et al. (2020), Zhang et al. (2022), Battauz et al. (2022b), and Battauz and Rotondi (2024). Other important instruments which exhibit early exercise rights are the so-called convertible bonds—they are studied by the same technique in Brennan and Schwartz (1977), Tsiveriotis and Fernandes (1998), Ayache et al. (2003) and De Spiegeleer et al. (2011). On the other hand, the variety of financial risks lead to the necessity of different novel instruments and methods for their evaluation. This conclusion is supported by the large number of modifications that appeared in the recent years—American power options (Miao et al. 2016; Lee 2020), American strangles (Jeon and Oh 2019; Qiu 2020) and (Jeon and Kim 2022), quanto options (Battauz et al. 2022a), CoCo bonds (Milanov et al. 2019), CoCoCats (Burnecki et al. 2019), etc.

This growing literature motivated us to investigate the American style derivatives under a general framework. At the same time the focus of this paper is on the power payment structures. However, we impose much weaker conditions which guarantee that pricing of such derivatives leads to one-sided optimal stopping problems. This means that the state space can be divided into two subsets of the forms $(0, c(t))$ and $(c(t), \infty)$—in one of them the immediate exercise is the optimal strategy, whereas in the other keeping the asset leads to a better result. If the set $(0, c(t))$ consists of the optimal points, then we name the derivative put-style. Otherwise, if the optimal set is $(c(t), \infty)$, then the derivative is call-style. This distinction is made by analogy with the ordinary American options. The boundary between the sets, namely $c(t)$, is called early exercise or optimal boundary. We have two degenerate cases, $c(t) \equiv 0$ and $c(t) \equiv \infty$. Such derivatives exhibit a put feature as well as call. However, more or less these cases are trivial—the immediate exercise is always or never optimal.

The assumption we made is that the underlying asset is driven by a log-normal process. We add to the payoff an additional discount factor which has an outstanding importance. First, it is the unique deterministic factor which makes early exercising preferable. But the more important fact is that this factor can be viewed as a dividend rate. In such a way, we can consider assets paying continuously dividends without introducing new variables.

We solve the arising one-sided optimal stopping problem in several steps. First, we prove several propositions for the shape of the optimal boundary. It turns out that it is an increasing function w.r.t. the time to maturity for the call-style derivatives and, on the opposite, it decreases for the put-ones. The next step is to derive the endpoints of the boundary—the initial one corresponds to the maturity date whereas the infinity value is for the perpetual derivative. Thereafter, we approximate the whole boundary using an exponent of piecewise linear functions. On every step we maximize the financial utility of the derivative's holder. Once we derive the boundary, the arising free boundary equation turns into a boundary value problem in a known region. We adapt the Crank–Nicolson finite difference approach to it and that way we derive the fair price.

The main restriction we impose is that the payoff is a twice differentiable function. Obviously, this condition is not satisfied by the most traded financial derivatives, namely the options—their payoffs are not differentiable at the strike. We can overcome this problem approximating the payoffs by suitable twice differentiable functions. This is possible, because the non-differentiability appears only at a single point, namely the strike. We provide as an example an approach for evaluating classical options.

As we mentioned above, the power payoff functions of the type $Mx^n + K$ have a center place in our study. They generate derivatives which can be viewed as power futures contracts—long and short positioned. The choice of the power functions is motivated by several circumstances. The main of them is that these payoffs allow the investors to hedge different non-linear risks. For example, an investor will prefer derivatives with larger values of the power n when she needs to hedge stronger the deeply in-the-money positions. On the opposite, the lower values of n are preferable for the near-the-money positions. The terms *in-the-money, near-the-money, etc.* are used analogously to the meaning of the option positions, i.e. these terms indicate the payoff sign. In the usual case, i.e. $n = 1$, the values $M = 1$ and $K < 0$ leads to a long positioned futures related to a call option. Analogously, the short futures contracts ($M = -1$ and $K > 0$) are related to put options. It turns out that the derivative's style does not depend only on the signs of the constants M and K when $n \neq 1$, but it depends on the sign of another constant too. Some parameters' values lead again to the degenerated derivatives for which the immediate exercise is always or never optimal. We examine all cases in detail. After that we present some numerical experiments which illustrate the derived results. As further work, we can consider power options which payoff can be defined as $(Mx^n + K)^+$.

The paper is organized as follows. We present the base we use later in Sect. 2. The shape of the optimal regions is obtained in Sect. 3. We examine the perpetual derivatives in Sect. 4. We present in Sect. 5 numerical algorithms for deriving the optimal boundary and the price under the finite maturity horizon. We examine power payoffs and the related derivatives in Sect. 6. The American call options are discussed in the light of our framework in Sect. 7.

2 Preliminaries

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, Q)$ be a filtered probability space satisfying the usual conditions—the filtration is right continuous and complete—and B_t be a Brownian motion under this space. The measure Q is assumed to be risk neutral. Hence, working in the Black and Scholes (1973) framework, the underlying asset follows the log-normal dynamics

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t\mathrm{d}B_t. \tag{2.1}$$

We assume that the risk-free rate is the constant r and the additional discount factor is $\lambda \geq 0$. We admit negative values for the risk-free rate, but we impose positiveness of the total discount rate, $r + \lambda > 0$. The following proposition explains why we can consider dividend paying assets in this framework.

Proposition 1 *If δ is the dividend rate, then a (r, λ, δ)-model is equivalent to the $(r - \delta, \lambda + \delta, 0)$-model.*

Proof See Proposition 2.3 from Zaevski (2020a). □

Let the derivative maturates at a moment $T \leq \infty$. Let us denote by $N(t, x)$ the payoff of the derivative, i.e. if the holder exercises the instrument at the moment $t \leq T$ at the spot price $S_t = x$, then he/she receives amount of $N(t, x)$. We also assume that the time dependence is presented only by the additional discount rate λ. Thus we have the presentation

$$N(t, x) = e^{-\lambda t}G(x) \tag{2.2}$$

for some twice differentiable function $G(x)$. Let us denote by $\mathcal{A}$ the infinitesimal generator of process (2.1), i.e. for the twice differentiable function $f(\cdot)$, $(\mathcal{A}f)(x)$ is

$$(\mathcal{A}f)(x) = rxf'(x) + \frac{\sigma^2x^2}{2}f''(x).$$

We define also the following related differential operator over the C^2 functions

$$(\mathcal{B}g)(x) = (\mathcal{A}g)(x) - (r + \lambda) g(x). \tag{2.3}$$

Since the underlying asset is modeled by a Markov process we can work w.r.t. the time to maturity $\tau = T - t$ instead of the original time. This means that we can assume $t = 0$ when this will lead to a simplification. We shall use the notation (t, x) w.r.t. the original parametrization and $(x; \tau)$ w.r.t. the time to maturity.

The following definition presents the derivatives we investigate in this study.

Definition 1 Let the derivative is characterized by payoff (2.2).

1. We name the instrument a call-style derivative if the following condition holds. If $(\mathcal{B}G)(x) < 0$ for some x, then $(\mathcal{B}G)(y) < 0$ for all $y \geq x$.
2. The condition for the put-style derivatives is as follows: if $(\mathcal{B}G)(x) < 0$ for some x, then $(\mathcal{B}G)(y) < 0$ for all $y \leq x$.

Although this definition seems artificial from a financial point of view, it has its significance—practical and theoretical. One of the main properties of the classical options is the fact that the immediate exercise is preferable for the holder if the asset price is large (for the calls) or small (for the puts) enough. The introduced above conditions guarantee keeping this property. Also, they are not so restrictive.

We shall use the symbol P hereafter for the derivative price, i.e. $P^p(t, x)$ and $P^c(t, x)$ are the prices of put- and call-style derivatives assuming that the price of the underlying asset at the moment t is x. If we have a parametrization w.r.t. the time to maturity we shall use the notations $P^p(x; \tau)$ and $P^c(x; \tau)$.

Remark 1 Let us explain the meaning of Definition 1. The payoff of an American call option with strike K is $N(t, x) = e^{-\lambda t}(x - K)^+$ or respectively $G(x) = (x - K)^+$. For the put options we have $G(x) = (K - x)^+$. Note that these functions are not differentiable at the point $x = K$. Nevertheless, we can examine functions $\overline{G}(x) = x - K$ and $\overline{G}(x) = K - x$ having in mind that early exercising for American options is never optimal when $\overline{G}(x) < 0$. Thus differential operator (2.3) turns $\left(\mathcal{B}\overline{G}\right)(x) = -\lambda x + (r + \lambda) K$ for call options and $(\mathcal{B}G)(x) = \lambda x - (r + \lambda) K$ for put ones. Thus the conditions of Definition 1 are satisfied.

3 Exercise regions

The next step is to divide the state space which contains the points (t, x) for $t \in [0, T]$ and $x \in (0, +\infty)$. The points for which the immediate exercise is the best strategy for the derivative's owner form the so-called optimal set—we denote it by Υ. The rest points form the continuation set, i.e. there exists a strategy which expected financial result is larger. We shall denote this set by $\overline{\Upsilon}$, i.e $\overline{\Upsilon} = \{[0, T] \times (0, +\infty)\} / \Upsilon$. We shall use also the notations $\Upsilon_{t,T}$ and $\overline{\Upsilon}_{t,T}$ for the corresponding sets at a fixed moment t assuming that the derivative maturates at the moment T. Alternatively, we use $\mathcal{T}_\tau$, Υ_τ, and $\overline{\Upsilon}_\tau$ instead $\mathcal{T}_{[t,T]}$, $\Upsilon_{t,T}$, and $\overline{\Upsilon}_{t,T}$ for the corresponding sets if we parametrize w.r.t. the time to maturity. All these is mathematically formalized in the following definition.

Definition 2 Let the set $\mathcal{T}_{[t,T]}$ consists of all stopping times with values between t and T. Also let us denote by $S_u^{t,x}$, $u \geq t$ the underlying asset assuming that its value at time t is x. If $t = 0$, we simplify the notation to S_u^x. Also, we denote by $E^{t,x}$ and E^x the corresponding expectations.

1. The point (t, x) is optimal, $(t, x) \in \Upsilon$, if for every stopping time $\zeta \in \mathcal{T}_{[t,T]}$,

$$N(t, x) \geq E^{t,x}\left[e^{-r(\zeta - t)} N\left(\zeta, S_\zeta\right)\right].$$

2. Otherwise, $(t, x) \in \overline{\Upsilon}$ if there exists a stopping time $\zeta \in \mathcal{T}_{[t,T]}$, such that

$$N(t, x) < E^{t,x}\left[e^{-r(\zeta - t)} N\left(\zeta, S_\zeta\right)\right]. \tag{3.1}$$

The following proposition characterizes the possible exercising moments.

Proposition 2 *A necessary condition a point (t, x) to be optimal, $(t, x) \in \Upsilon$, is $(\mathcal{B}G)(x) < 0$.*

Proof Suppose that at the moment t the asset price is $S_t = x$ and $(\mathcal{B}G)(x) \geq 0$. Let the investor's strategy be the exercise after an infinitely small period ϵ. It leads to a financial result $E^{t,x}\left[e^{-(r+\lambda)(t+\epsilon)} G\left(S_{t+\epsilon}\right)\right]$ which is larger than the result of immediate exercising since

$$\lim_{\epsilon \to 0} \frac{E^{t,x}\left[e^{-(r+\lambda)(t+\epsilon)} G\left(S_{t+\epsilon}\right)\right] - e^{-(r+\lambda)t} G(x)}{\epsilon} = e^{-(r+\lambda)t}\left(\mathcal{B}G\right)(x) \geq 0.$$

□

Next we obtain the shape of the optimal regions proving several propositions.

Proposition 3 *If $\tau_1 > \tau_2$ and the point $(x; \tau_1)$ is optimal for a derivative (no matter put- or call-style), then the point $(x; \tau_2)$ is optimal too.*

Proof Suppose that $(x; \tau_2) \notin \Upsilon$ and thus there exists a strategy ζ such that inequality (3.1) holds. But this contradicts to $(x; \tau_1) \in \Upsilon$ because $T_{\tau_1} \supset T_{\tau_2}$ since $\tau_1 > \tau_2$. □

Proposition 4 *Let $(x; \tau)$ be an optimal point, $(x; \tau) \in \Upsilon_\tau$.*

1. *If the derivative is a call-style and $y \geq x$, then $(y; \tau) \in \Upsilon_\tau$ too.*
2. *If the derivative is put-style and $y \leq x$, then $(y; \tau) \in \Upsilon_\tau$.*

Proof Suppose that for a call-style derivative $(x; \tau) \in \Upsilon_\tau$ and $y > x$. Let $\overline{\zeta}$ be an arbitrary stopping time, ζ^x be the first hitting moment of the underlying asset starting at the point y to the value x, and $\zeta = \overline{\zeta} \wedge \zeta^x \wedge \tau$. Note that the strategy ζ gives a not worse financial result then $\overline{\zeta}$ since the underlying asset is driven by a Markov process and $(x; \overline{\tau})$ is an optimal point for every $\overline{\tau}$ such that $\overline{\tau} < \tau$ due to Proposition 3. Also let us mention that $(\mathcal{B}G)\left(S_u(\omega)\right) < 0$ for every u such that $u < \zeta(\omega)$ because for such sample paths we have $S_u(\omega) > x$ and hence we can use the definition of a call-style derivative—the first point from Definition 1. Using the Dynkin's formula we derive

$$\begin{aligned} E^y\left[e^{-(r+\lambda)\overline{\zeta}} G\left(S_{\overline{\zeta}}\right)\right] - G(y) &\leq E^y\left[e^{-(r+\lambda)\zeta} G\left(S_\zeta\right)\right] - G(y) \\ &= E^y\left[\int_0^\zeta (\mathcal{B}G)(S_u)\right] < 0. \end{aligned} \tag{3.2}$$

Therefore the point $(y; \tau)$ is optimal because inequality (3.2) is true for an arbitrary stopping time $\overline{\zeta}$. The proof for a put-style option is analogous. □

The shapes of the optimal regions can be obtained through Proposition 4. We present them in the following corollary.

Corollary 1 1. *The optimal region of a call-style derivative consists of all points above some boundary—the boundary is known as the optimal boundary.*
2. *The region for a put-style derivative consists of all positive points below its optimal boundary.*

Remark 2 Note that these boundaries may be infinitely large or zero valued. A well-known example is the call options when the additional discounting is missing, $\lambda = 0$. This way the derivatives exhibit jointly put and call features.

The behavior of the optimal boundaries can be obtained in a way similar to one presented in Jacka (Jacka 1991, proposition 2.2). We shall only sketch the proof.

Proposition 5 *The following statements hold.*

1. *The optimal boundary for a call-style derivative is an increasing function w.r.t. the time to maturity.*
2. *The boundary of a put-style derivative decreases w.r.t. the time to maturity.*

Proof Let us consider a call-style derivative and a point at the optimal boundary $(\tau, c(\tau))$. Suppose that at this point the boundary is a decreasing function w.r.t. the time to maturity. This means that there exists $\epsilon > 0$ such that $(\epsilon, c(\tau)) \notin \Upsilon$ since the boundary is a continuous function.[1] This contradicts to the Propositions 3 and 4. The rest of the proof can be made in the analogous way. □

The next step is to determine the initial values of the optimal boundaries:

Proposition 6 *The following statements hold for the limits of the optimal boundaries when the time to maturity tends to zero.*

1. *Let us have a call-style derivative and B_1 be defined as*

$$B_1 = \inf\{x : (\mathcal{B}G)(x) < 0\}.$$

Note that this value exists, despite it can be the infinity, due to the first statement of Definition 1. Namely B_1 is the value of the optimal boundary when the time to maturity is zero, $c(0) = B_1$.

2. *If the derivative is put-style and A_1 is defined as*

$$A_1 = \sup\{x : (\mathcal{B}G)(x) < 0\},$$

then $c(0) = A_1$.

Proof Let us consider first a call-style derivative. We have that the initial boundary value is not less than B_1, $c(0) \geq B_1$, due to Proposition 2. It left to be proven that all points above B_1 are optimal. Suppose the opposite, i.e. some point $x > B_1$ is not optimal near the initial moment $\tau = 0$. Using the inequality $P^c(t,x) > N(t,x)$, which holds in the continuation region, together with the Black–Scholes equation which is satisfied in this region, we obtain

$$\begin{aligned}
0 &< \lim_{t\to T} \frac{P^c(t,x) - N(t,x)}{T-t} \\
&= -\lim_{t\to T} \frac{P^c(T,x) - P^c(t,x)}{T-t} + \lim_{t\to T} \frac{N(T,x) - N(t,x)}{T-t} \\
&= \mathcal{A}P^c(T,x) - rP^c(T,x) + N_t(T,x) \\
&= e^{-\lambda t}\mathcal{A}G(x) - re^{-\lambda t}G(x) - \lambda e^{-\lambda t}G(x) \\
&= e^{-\lambda t}\mathcal{B}G(x) < 0.
\end{aligned}$$

[1] The last is true because the function $G(x)$ is twice differentiable.

The contradiction finishes the proof for the call-style derivatives. The result for put-style instruments can be obtained in the similar manner. □

4 Perpetual derivatives

We now consider derivatives without maturity constraints, i.e. $\tau = T = \infty$. In this case the optimal boundaries are flat—we denote them by A_2 and B_2 for put- and call-style derivatives, respectively. Let us denote by ζ^c the first hitting moment of the underlying asset to the value c. Note that ζ^c can be viewed also as the Brownian motion's first hitting to the linear function

$$d(t) = d_1 t + d_2, \tag{4.1}$$

where d_1 and d_2 are

$$\begin{aligned} d_1 &= \frac{\sigma}{2} - \frac{r}{\sigma} \\ d_2 &= \frac{1}{\sigma} \ln\left(\frac{c}{x}\right). \end{aligned} \tag{4.2}$$

The stopping time ζ^c is also the first hitting of the Brownian motion with drift $\mu = -d_1$ to the value d_2. We need the following reported in Borodin and Salminen (2015, page 223 (2.0.1)) formulas.

Proposition 7 *If ζ is the first hitting time of a Brownian motion with drift μ to the level a, then*

$$\begin{aligned} E\left[e^{-y\zeta} I_{\zeta<\infty}\right] &= e^{-\left(\sqrt{\mu^2+2y}-\mu\right)a}, \quad \text{if } a > 0 \\ E\left[e^{-y\zeta} I_{\zeta<\infty}\right] &= e^{\left(\sqrt{\mu^2+2y}+\mu\right)a}, \quad \text{if } a < 0. \end{aligned} \tag{4.3}$$

4.1 Call-style derivatives

Suppose first that the derivative is call-style. We need the following lemmas.

Lemma 1 *Let the constant γ_c and the function $g_c(c)$ be defined as*

$$\begin{aligned} g_c(c) &= \frac{G(c)}{c^{\gamma_c}} \\ \gamma_c &= \frac{\sqrt{d_1^2 + 2(r+\lambda)} + d_1}{\sigma} \\ &= \sqrt{\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r+\lambda}{\sigma^2}} - \left(\frac{r}{\sigma^2} - \frac{1}{2}\right). \end{aligned} \tag{4.4}$$

The financial result of the strategy ζ^c, assuming that the asset starts from a point $x < c$ is

$$P^c(x; c) = g_c(c)\, x^{\gamma_c} + \lim_{T\to\infty} E^x\left[e^{-(r+\lambda)T} G(S_T)\, I_{T<\zeta^c}\right]. \tag{4.5}$$

Proof Equation (4.3) from Proposition 7 leads to

$$
\begin{aligned}
P^c(x;c) &= E^x\left[e^{-(r+\lambda)\zeta^c} G\left(S_{\zeta^c}\right) I_{\zeta^c<\infty}\right] + \lim_{T\to\infty} E^x\left[e^{-(r+\lambda)T} G(S_T)\, I_{T<\zeta^c}\right] \\
&= G(c)\, E^x\left[e^{-(r+\lambda)\zeta^c} I_{\zeta^c<\infty}\right] + \lim_{T\to\infty} E^x\left[e^{-(r+\lambda)T} G(S_T)\, I_{T<\zeta^c}\right] \\
&= G(c)\exp\left\{-\frac{\sqrt{d_1^2+2(r+\lambda)}+d_1}{\sigma}\ln\left(\frac{c}{x}\right)\right\} \\
&\quad + \lim_{T\to\infty} E^x\left[e^{-(r+\lambda)T} G(S_T)\, I_{T<\zeta^c}\right] \\
&= g_c(c)\, x^{\gamma_c} + \lim_{T\to\infty} E^x\left[e^{-(r+\lambda)T} G(S_T)\, I_{T<\zeta^c}\right].
\end{aligned}
$$

□

An immediate consequence of Corollary 1 is the following proposition.

Proposition 8 *If the maximum of the function $P^c(x;c)$, defined in Eq. (4.5), w.r.t. the variable c in the interval $[x,\infty)$ is at a point $c(x)$, then the maximum of the function $P^c(y;c)$ for every y, $x<y<c(x)$, is again at the point $c(x)$.*

Suppose now that x is small enough. The lemmas below characterize the behavior at the optimal point.

Lemma 2 *If the function $P^c(x;c)$ defined in Eq. (4.5) has a local maximum in a point $\overline{c}\in(x,\infty)$, then $(\mathcal{B}G)(\overline{c})\le 0$.*

Proof Suppose that $(\mathcal{B}G)(\overline{c})>0$. Therefore there exists a small enough but larger than $\overline{c}$ constant $\tilde{c}$ such that $(\mathcal{B}G)(c)>0$ for all $c<\tilde{c}$. The Dynkin's formula leads to

$$
\begin{aligned}
P^c(x;c) &= \lim_{T\to\infty} E^x\left[e^{-(r+\lambda)\zeta^c\wedge T} G\left(S_{\zeta^c\wedge T}\right)\right] \\
&= G(x) + \lim_{T\to\infty} E^x\left[\int_0^{\zeta^c\wedge T} (\mathcal{B}G)(S_u)\,\mathrm{d}u\right].
\end{aligned}
\tag{4.6}
$$

Using the fact that the function $P^c(x;c)$ has a local maximum in the point $\overline{c}$ we conclude that $P^c(x;\overline{c})>P^c(x;\tilde{c})$ for all $x<\overline{c}$. Having in mind Eq. (4.6) and taking the limit $x\uparrow\overline{c}$ we conclude

$$
\begin{aligned}
G(\overline{c}) &= \lim_{x\uparrow\overline{c}} P^c(x;\overline{c}) > \lim_{x\uparrow\overline{c}} P^c(x;\tilde{c}) = P^c(\overline{c};\tilde{c}) \\
&= G(\overline{c}) + \lim_{T\to\infty} E^{\overline{c}}\left[\int_0^{\zeta^{\tilde{c}}\wedge T} (\mathcal{B}G)(S_u)\,\mathrm{d}u\right].
\end{aligned}
$$

We finish the proof having in mind that $(\mathcal{B}G)(S_u)$ is positive at all sample paths before the stopping time $\zeta^{\tilde{c}}$. □

Lemma 3 *The function $P^c(x;c)$ defined in Eq. (4.5) has no more than one local maximum in the interval (x,∞). This way its global maximum is either this local one or one of the interval endpoints.*

Proof Suppose the opposite, i.e. there exist at least two local maximums—we denote them by c_1 and c_2. Note that $(\mathcal{B}G)\left(c_{1,2}\right) < 0$ due to Lemma 2. Also, the function $P^c(x; c)$ is increasing in some sub-interval $(C_1, C_2) \subset (c_1, c_2)$. Let us consider an initial point $x \in (C_1, C_2)$ and a strategy of exercising at the first hitting to $x + \delta$, where with δ we denote an infinitely small positive value. Having in mind the form of formula (4.6), the increasing behavior at the point x, and the Dynkin's formula, we see that $(\mathcal{B}G)(x) > 0$—the proof of this fact is similar to the proof of Lemma 2. But this contradicts to $(\mathcal{B}G)(c_1) < 0$, $c_1 < C_1 < x$, and the first statement of Definition 1. □

The proven above statements (Proposition 8 and Lemmas 2 and 3) lead to the following proposition.

Proposition 9 1. *If there exists an initial point x, for which x is strictly less than $c(x)$, $x < c(x)$, then $c(x)$ is the optimal boundary.*
2. *If $x = c(x)$ for all x, then the optimal boundary is the zero, i.e. all points are optimal.*
3. *If $x < c(x)$ for every x, then the optimal boundary is the infinite which means that early exercising is never optimal.*

Proposition 9 shows the way we derive the optimal boundary. We chose a small enough x, for which $x < c(x)$. Thus the optimal boundary is namely $c(x)$. If such x does not exist, then the optimal boundary is the zero.

We can formulate now our main result for pricing perpetual call-style derivatives.

Theorem 1 *The optimal boundary of a perpetual American call-style derivatives is*

$$B_2 = \lim_{x \to 0} \arg\max_{c \in [x, \infty)} \left\{P^c(x; c)\right\}.$$

If $x \geq B_2$, then the derivative price is $P^c(x; B_2) = G(x)$. Otherwise, if $x < B_2$, then the price is

$$P^c(x, B_2) = g_c(B_2)\, x^{\gamma_c} + \lim_{T \to \infty} E^x\left[e^{-(r+\lambda)T} G(S_T)\, I_{T < \zeta^{B_2}}\right], \tag{4.7}$$

where the function $g_c(\cdot)$ and the constant γ_c are defined in Eq. (4.4). Note that if $B_2 = \infty$, then formula (4.7) turns to

$$P^c(x; \infty) = \lim_{c \to +\infty}\left[\frac{G(c)}{c^{\gamma_c}} x^{\gamma_c} + \lim_{T \to \infty} E^x\left[e^{-(r+\lambda)T} G(S_T)\, I_{T < \zeta^c}\right]\right].$$

4.2 Put-style derivatives

We present below the analog of Theorem 1, when the derivative is put-style. The second statement of Corollary 1 shows that we have a lower first hitting problem.

Theorem 2 *Let the constant γ_p be defined as*

$$\begin{aligned}\gamma_p &= \frac{\sqrt{d_1^2 + 2(r+\lambda)} - d_1}{\sigma} \\ &= \sqrt{\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r+\lambda}{\sigma^2}} + \left(\frac{r}{\sigma^2} - \frac{1}{2}\right).\end{aligned} \tag{4.8}$$

The price of the derivative associated with the first hitting to a boundary below the starting point, $c < x$, is

$$P^p(x; c) = \frac{g_p(c)}{x^{\gamma_p}} + \lim_{T\to\infty} E^x\left[e^{-(r+\lambda)T} G(S_T) I_{T<\zeta^c}\right], \tag{4.9}$$

where the function $g_p(\cdot)$ is defined as

$$g_p(c) = G(c)\, c^{\gamma_p} \tag{4.10}$$

The function (4.9) has no more than one local maximum in the interval $(0, x]$. Let us denote by $c(x)$ the optimal boundary if the asset's initial value is x. The optimal boundary A_2 is obtained through the following statements:

1. *If $c(x) < x$ for some x, then $A_2 = c(x)$.*
2. *If $c(x) = x$ for all x's, then all points are optimal, i.e. $A_2 = \infty$.*
3. *If $c(x) < x$ for all x, then early exercising is never optimal i.e. $A_2 = 0$.*

If $0 < x \le A_2$, then the derivative price is just $P^p(x, A_2) = G(x)$. If $x > A_2 > 0$, then the price is

$$P(x; A_2) = \frac{g_p(A_2)}{x^{\gamma_p}} + \lim_{T\to\infty} E^x\left[e^{-(r+\lambda)T} G(S_T) I_{T<\zeta^{A_2}}\right].$$

If $A_2 = 0$, then

$$P(x; 0) = \lim_{c\to 0}\left[\frac{g_p(c)}{x^{\gamma_p}} + \lim_{T\to\infty} E^x\left[e^{-(r+\lambda)T} G(S_T) I_{T<\zeta^c}\right]\right].$$

Proof Suppose that the initial asset value x is large enough and the derivative's holder exercises at the value $c < x$. Having in mind the second statement of Proposition 7 we derive for the put-price

$$\begin{aligned}
P^p(x; c) &= E^x\left[e^{-(r+\lambda)\zeta^c} G\left(S_{\zeta^c}\right) I_{\zeta^c<\infty}\right] + \lim_{T\to\infty} E^x\left[e^{-(r+\lambda)T} G(S_T) I_{T<\zeta^c}\right] \\
&= G(c)\, E^x\left[e^{-(r+\lambda)\zeta^c} I_{\zeta^c<\infty}\right] + \lim_{T\to\infty} E^x\left[e^{-(r+\lambda)T} G(S_T) I_{T<\zeta^c}\right] \\
&= G(c) \exp\left\{\frac{\sqrt{d_1^2 + 2(r+\lambda)} - d_1}{\sigma} \ln\left(\frac{c}{x}\right)\right\} \\
&\quad + \lim_{T\to\infty} E^x\left[e^{-(r+\lambda)T} G(S_T) I_{T<\zeta^c}\right] \\
&= \frac{g_p(c)}{x^{\gamma_p}} + \lim_{T\to\infty} E^x\left[e^{-(r+\lambda)T} G(S_T) I_{T<\zeta^c}\right].
\end{aligned}$$

The rest of the proof is identical to the call case. □

5 Finite maturity case

Assume now that the maturity is finite, i.e. $T < \infty$. Let us exclude the combined put-call derivatives. We make this without loss of generality, because the combined cases lead to derivatives for which early exercising is always or never optimal. The following reasons confirm this. If the derivative exhibits a combined feature, then $\mathcal{B}(G)(x) \ge 0$ for all x's or $\mathcal{B}(G)(x) \le 0$ ever. The put and call initial boundaries are $A_1 = 0$ and $B_1 = \infty$ in the first case. Proposition 5 shows that early exercising is never optimal. For the second case we can examine the derivative with the opposite payoff.

5.1 Optimal boundary

The first our task is to approximate the optimal boundary. Let us divide the time interval $[0, T]$ into l sub-intervals, namely $0 \equiv t_0 < t_1 < \cdots < t_l \equiv T$. Let the owner's strategy ζ be the first hitting moment to the level $\exp(a_i t + b_i)$ if it happens in the interval $\left[t_{i-1}, t_i\right), i = 1, 2, \ldots, l$. We also impose a continuity at the nodes—$\exp(a_i t_i + b_i) = \exp(a_{i+1} t_i + b_{i+1}) \equiv c_i$. Let us consider a European style derivative which expires at the moment $\zeta \wedge T$ with payoff $\exp(-\lambda t)\, G(t)$. We denote its price by

$$P(x; \{t_0, \ldots, t_l\}; \{c_0, \ldots, c_l\}) = E^x\left[e^{-(r+\lambda)(\zeta \wedge T)} G\left(S_{\zeta \wedge T}\right)\right]. \tag{5.1}$$

Our algorithm for deriving the optimal boundary is as follows.

1. The value of the exercise boundary at the maturity, c_l, is given in Proposition 6, the first statement for calls and the second one for puts.
2. Suppose that we have derived all boundary values after some index $m \leq l$, namely c_m, c_{m+1},…, c_l. The call boundary at the previous node is obtained as the lower value of x for which the payment $h(c) = P(x; \{0, t_m - t_{m-1}, \ldots, t_l - t_{m-1}\}; \{c, c_m, \ldots, c_l\})$ given in Eq. (5.1) achieves its maximum in the interval $[x, \infty)$ for $c = x$. This is the lower value which makes the immediate exercise optimal. For put-style derivatives our approximation is the higher value of x for which the maximum in the interval $(0, x]$ is for $c = x$.

5.2 Pricing

Once we approximate the optimal boundary we can obtain the derivative price viewing it as the solution of a boundary value problem (BVP, hereafter). We begin with a call-style derivative. The region for the BVP is $(t, x) \in \{(0, T) \times (0, c(t))\}$. If the initial point is outside this range, then the price is simply $e^{-\lambda t} G(x)$. The BVP can be written as

$$\begin{aligned}
&F_t(t, x) + r x F_x(t, x) + \frac{1}{2}\sigma^2 x^2 F_{xx}(t, x) - r F(t, x) = 0\\
&F(t, 0) = e^{-r(T-t)} e^{-\lambda T} G(0), \quad t \in (0, T)\\
&F(t, c(t)) = e^{-\lambda t} G(x), \quad t \in (0, T)\\
&F(T, x) = e^{-\lambda T} G(x), \quad x \in (0, B_1).
\end{aligned} \tag{5.2}$$

We use the following modification of the Crank–Nicolson finite difference method.

1. We create uniformly the time and space grids respectively with M and N nodes—$T \equiv t_1 > t_2 > \cdots > t_M \equiv 0$ and $0 \equiv x_1 < x_2 < \cdots < x_N \equiv c(0)$. The derivative price at the (m, n)th node shall be denoted by $F(m, n)$. The values $F(1, n)$ and $F(M, n)$ are the prices at the maturity and in the initial moment, respectively, because we work backward.
2. We approximate the boundary $c(t)$ at $\overline{M} \ll M$ points, $C_1, C_2,\ldots,C_{\overline{M}}$, using the algorithm presented in Sect. 5.1 and then interpolate the whole boundary by cubic splines.
3. The terminal condition leads to
$$F(1, n) = e^{-\lambda T} G(x_n).$$
4. Let us denote by k_m the lowest n such that $x_{k_m} > C_m$.

5. The lower and upper boundary conditions are integrated by

$$F(m,1) = e^{-r(T-t_m)} e^{-\lambda T} G(0)$$
$$F(m,n) = e^{-\lambda t_m} G(x_n) \quad \forall\, m \text{ and } n \geq k_m.$$

6. We derive the values of $F(m,n)$ iteratively using the already obtained $F(i,n)$ for all n and $i < m$ by the Crank–Nicolson scheme. The corresponding derivatives are approximated by formulas (C.1) and the BVP (5.2) can be presented as Eq. (C.2). Rearranging w.r.t. $n \in \{2, \ldots, k_m - 1\}$ we reach a linear system for $F(m,n)$ given in Eqs. (C.3), (C.4), and (C.5).

We need a little adjustment of the pricing algorithm if we have a put-style derivative. The main difference is that the continuation region is infinite above. For this we set an auxiliary large enough boundary—we denote it by $\overline{C}$. A relatively good approximation of the price function at this boundary is the value of the corresponding European style derivative. Alternatively, we can use a Monte Carlo simulation. Let us denote this approximation by $P^E\left(t; \overline{C}\right)$. Thus the BVP (5.2) turns to

$$\begin{aligned}
&F_t(t,x) + rxF_x(t,x) + \frac{1}{2}\sigma^2 x^2 F_{xx}(t,x) - rF(t,x) = 0 \\
&F\left(t,\overline{C}\right) = P^E\left(t;\overline{C}\right), \quad t \in (0,T) \\
&F(t, c(t)) = e^{-\lambda t} G(x), \quad t \in (0,T) \\
&F(T,x) = e^{-\lambda T} G(x), \quad x \in \left(A_1, \overline{C}\right).
\end{aligned} \tag{5.3}$$

We can easily adapt the presented above Crank–Nicolson finite difference approach to BVP (5.3).

6 Power payoff functions

Suppose now that the payoff $G(\cdot)$ is given by $G(x) = Mx^n + K$, where $M \in \{-1, 1\}$, n is a positive number, and K an arbitrary real number. Note that the case $n < 0$ can be considered in our framework too because the following presentation holds

$$\frac{1}{S_t} = \frac{1}{x} e^{\left(\overline{r} - \frac{\sigma^2}{2}\right)t + \sigma \overline{B}_t},$$

where $\overline{r} = -r + \sigma^2$ and $\overline{B}_t = -B_t$ is again a Brownian motion.

We need first the following lemmas.

Lemma 4 *Let the constant L be defined as*

$$L = (n-1)\left(r + \frac{\sigma^2}{2}n\right) - \lambda. \tag{6.1}$$

If $L > 0$, then $\gamma_c < n$ and $\gamma_p < n + 2\frac{r}{\sigma^2} - 1$ and vice versa. If $L = 0$, then the inequalities turn to equalities.

Proof Suppose that $L \geq 0$ or equivalently

$$\lambda \leq (n-1)\left(r + \frac{\sigma^2}{2}n\right). \tag{6.2}$$

Thus inequality $r + \lambda > 0$ leads to

$$2\frac{r}{\sigma^2} + n - 1 > 0, \tag{6.3}$$

since

$$0 < r + \lambda \leq n\frac{\sigma^2}{2}\left(2\frac{r}{\sigma^2} + n - 1\right).$$

Combining formulas (6.2) and (4.4) we derive

$$\gamma_c \leq \sqrt{\left(\frac{r}{\sigma^2} - \frac{1}{2} + n\right)^2} - \left(\frac{r}{\sigma^2} - \frac{1}{2}\right). \tag{6.4}$$

Inequality (6.3) shows that

$$\frac{r}{\sigma^2} - \frac{1}{2} + n = \frac{1}{2}\left(2\frac{r}{\sigma^2} + n - 1\right) + \frac{n}{2} > 0. \tag{6.5}$$

Hence Eq. (6.4) indeed leads to $\gamma_c \leq n$. Suppose now that $L < 0$ or equivalently

$$\lambda > (n-1)\left(r + \frac{\sigma^2}{2}n\right).$$

Therefore

$$\begin{aligned} \gamma_c &> \sqrt{\left(\frac{r}{\sigma^2} - \frac{1}{2} + n\right)^2} - \left(\frac{r}{\sigma^2} - \frac{1}{2}\right) \\ &\geq \left(\frac{r}{\sigma^2} - \frac{1}{2} + n\right) - \left(\frac{r}{\sigma^2} - \frac{1}{2}\right) = n. \end{aligned}$$

The statements for the constant γ_p follow from definitions (4.4) and (4.8). □

Lemma 5 *The functions $g_c(\cdot)$ and $g_p(\cdot)$ defined in Eqs.* (4.4) *and* (4.10) *are* $g_c(c) = \frac{Mc^n + K}{c^{\gamma_c}}$ *and* $g_p(c) = (Mc^n + K)\, c^{\gamma_p}$. *Their derivatives are*

$$\begin{aligned} g_c^{'}(c) &= \frac{Mc^n(n-\gamma_c) - K\gamma_c}{c^{\gamma_c+1}} \\ g_p^{'}(c) &= c^{\gamma_p - 1}\left[Mc^n\left(n+\gamma_p\right) + K\gamma_p\right]. \end{aligned}$$

Proof The proof is an immediate consequence of the form of the functions $g_c(\cdot)$ and $g_p(\cdot)$. □

Lemma 6 *If $L > 0$, $\theta = n\sigma$, and $k = L - \frac{\theta^2}{2}$, then $-\frac{\theta^2}{2} < k < \frac{d_1^2}{2} - \theta d_1$ and $d_1 < n\sigma$.*

Proof First, $k > -\frac{\theta^2}{2}$ since $L > 0$. Second, inequality $k < \frac{d_1^2}{2} - \theta d_1$ holds, because it is equivalent to

$$-(r+\lambda) < \frac{1}{2}\left(\frac{\sigma}{2} - \frac{r}{\sigma}\right)^2.$$

It left to check the inequality $d_1 < n\sigma$. Indeed, $L > 0$ leads to inequality (6.5) which is equivalent to $d_1 < n\sigma$. □

The next step is to obtain the initial value of the optimal boundary. We have for the operator $(\mathcal{B}G)(\cdot)$

$$\begin{aligned}(\mathcal{B}G)(x) &= Mx^n\left[(n-1)\left(r+\frac{\sigma^2}{2}n\right)-\lambda\right]-K(r+\lambda)\\ &= MLx^n - K(r+\lambda).\end{aligned} \tag{6.6}$$

We see from Eq. (6.6) that the derivative's style depends on the sign of the constants M, L, and K. Note that some cases lead to joint put-call features. For convenience, we shall assume that the derivative is call-style if $LM < 0$ and put-style otherwise. The case $L = 0$ is special—it corresponds to the undiscounted case ($\lambda = 0$) when the payoff is linear. We shall examine all these cases separately.

6.1 The limiting case $L = 0$

We have now a put-call derivative. Note that the equality in formula (6.2) holds since $L = 0$. Let us consider first the derivative as call-style. Suppose that $K \geq 0$. Formula (6.6) shows that $(\mathcal{B}G)(x) \leq 0$ and therefore $B_1 = 0$. Price function (4.5) consists of two parts. The second one can be derived as

$$\lim_{T\to\infty} E^x\left[e^{-(r+\lambda)T}G(S_T)I_{T<\zeta^c}\right] = Mx^n \lim_{T\to\infty} e^{-n^2\frac{\sigma^2}{2}T}E^x\left[e^{n\sigma B_T}\right]. \tag{6.7}$$

Note that inequality (6.5) is equivalent to $d_1 < n\sigma$. Hence, the third statement of Lemma 8 shows that limit (6.7) is zero. Therefore, price function (4.5) turns to

$$P^c(x;c) = g_c(c)\,x^n = \frac{Mc^n+K}{c^n}x^n, \tag{6.8}$$

because $\gamma_c = n$ due to Lemma 4. Having in mind Lemma 5 we see that $P^c(x;c)$ is a decreasing function w.r.t. the variable c. The second statement of Proposition 9 shows that $B_2 = 0$ too. Hence, the perpetual price is

$$P(x) = Mx^n + K. \tag{6.9}$$

Let us consider the derivative as put-style. We have now $A_1 = \infty$. Using the ninth statement of Lemma 8 (for $\theta = n\sigma$), Eq. (6.2) (note again that the inequality turns to equality when $L = 0$), and Lemma 4 we derive for price (4.9)

$$\begin{aligned}P^p(x;c) &= \frac{(Mc^n+K)c^{\gamma_p}}{x^{\gamma_p}} + \lim_{T\to\infty}E^x\left[e^{-(r+\lambda)T}\left(MS_T^n+K\right)I_{T<\zeta^c}\right]\\ &= \frac{(Mc^n+K)c^{\gamma_p}}{x^{\gamma_p}} + \lim_{T\to\infty}e^{-\frac{n^2\sigma^2}{2}T}E^x\left[e^{n\sigma B_T}I_{T<\zeta^c}\right]\\ &= \frac{(Mc^n+K)c^{\gamma_p}}{x^{\gamma_p}} + Mx^n\left(1-\left(\frac{c}{x}\right)^{n+\gamma_p}\right)\\ &= \frac{Kc^{\gamma_p}}{x^{\gamma_p}} + Mx^n.\end{aligned} \tag{6.10}$$

Hence, the value $c = x$ maximizes price (6.10). The second statement of Theorem 2 shows that $A_2 = \infty$ too which confirms that all points are optimal.

Let us consider now the case $K < 0$ and the derivative as a call one. Formula (6.6) leads $(\mathcal{B}G)(x) > 0$ and hence $B_1 = \infty$. Analogously we derive formula (6.14) for the price and using Lemma 5 we derive $B_2 = \infty$ too. If we consider the derivative as put-style, then

$A_1 = 0$. On the other hand, price (6.10) is a decreasing function w.r.t. the variable c and thus the third statement of Theorem 2 shows that $A_2 = 0$ or equivalently there are no optimal points. The perpetual price can be derived as the limit for $c \to 0$ in formula (6.10). Hence,

$$P(x) = Mx^n. \tag{6.11}$$

Note that the same result can be derived using call-price (6.8) for $c \to \infty$.

6.2 Call-style derivatives—*LM* < 0

Having in mind Eq. (6.6) we derive for the boundary value when time to maturity is zero

$$B_1 = \left(\max\left\{K\frac{r+\lambda}{ML}, 0\right\}\right)^{\frac{1}{n}}. \tag{6.12}$$

The next task is to derive the perpetual value. The second part of price function (4.5) now can be written as

$$\lim_{T\to\infty} E^x\left[e^{-(r+\lambda)T} G(S_T)\, I_{T<\zeta^c}\right] = Mx^n \lim_{T\to\infty} e^{\left(L-n^2\frac{\sigma^2}{2}\right)T} E^x\left[e^{n\sigma B_T}\right]. \tag{6.13}$$

Suppose first that $L < 0$ and $M = 1$. Thus we are in the first statement of Lemma 8 and therefore limit (6.13) is zero. Hence price function (4.5) is given by formula

$$P^c(x; c) = g_c(c)\, x^n = \frac{c^n + K}{c^{\gamma_c}} x^{\gamma_c}. \tag{6.14}$$

The inequality $L < 0$ and Lemma 4 leads to $\gamma_c > n$. Using Lemma 5 we see that the numerator of the derivative $g_c^{'}(c)$ is a decreasing function. If $K \geq 0$, then $g_c^{'}(c) \leq 0$ for $c > 0$ and therefore price function (4.5) decreases w.r.t. the variable c. We conclude that $B_2 = 0$ due to the second statement of Proposition 9. Note that $B_1 = 0$ too due to Eq. (6.12). This means that we have a combined derivative which perpetual price is given by formula (6.9).

If $K < 0$, then the derivative $g_c^{'}(c)\, 0$ has a root and $g_c^{'}(c) > 0$ before the root and $g_c^{'}(c) < 0$ after. Thus we conclude that namely this root is the perpetual value B_2. Hence

$$\begin{aligned} B_1 &= \left(K\frac{r+\lambda}{L}\right)^{\frac{1}{n}} \\ B_2 &= \left(-\frac{K\gamma_c}{\gamma_c - n}\right)^{\frac{1}{n}} \end{aligned} \tag{6.15}$$

and the perpetual price for $x < B_2$ is

$$P(x) = \frac{B_2^n + K}{B_2^{\gamma_c}} x^{\gamma_c}. \tag{6.16}$$

Suppose now that $L > 0$ and $M = -1$. Having in mind Lemma 8 and Eq. (6.13) we can see that the constants k and θ from this lemma are $k = L - \frac{\theta^2}{2}$ and $\theta = n\sigma$. Lemma 6 shows that the sixth statement of Lemma 8 is actual and therefore limit (6.13) is zero. Hence, the price function is given again by formula (6.14). Using Lemma 4 we see that inequality $L > 0$ leads to $\gamma_c < n$. We see that the same conclusion w.r.t. the sign of K are valid, because now

$M = -1$. The only difference is in the sign in the formulation of boundaries (6.15) which turn to

$$\begin{aligned} B_1 &= \left(-K\frac{r+\lambda}{L}\right)^{\frac{1}{n}} \\ B_2 &= \left(-\frac{K\gamma_c}{n-\gamma_c}\right)^{\frac{1}{n}}. \end{aligned} \tag{6.17}$$

Perpetual price (6.16) turns to

$$P(x) = \frac{-B_2^n + K}{B_2^{\gamma_c}} x^{\gamma_c}. \tag{6.18}$$

6.3 Put-style derivatives—*LM* > 0

Equation (6.6) shows that the initial boundary value A_1 is given again by formula (6.12). We have to derive the perpetual value. Suppose first that $L < 0$ and $M = -1$. The second part of price function (4.9) can be written as formula (6.13). The first statement of Lemma 8 shows that limit (6.13) is zero which means that price function (4.9) is given by

$$P^p(x; c) = \frac{g_p(c)}{x^{\gamma_p}} = \frac{(-c^n + K)\, c^{\gamma_p}}{x^{\gamma_p}}. \tag{6.19}$$

Lemma 5 shows that the second part of the derivative $g_p{}'(c)$, namely $-c^n\left(n+\gamma_p\right)+K\gamma_p$, is a decreasing function. If $K \le 0$, then $g_c^{'}(c) \le 0$ for all admissible values of c and therefore price function (4.5) decreases w.r.t. the variable c. We conclude that $A_2 = 0$ due to the third statement of Theorem 2. Note that $A_1 = 0$ too because of Eq. (6.12). Hence, the perpetual price is $P(x) = 0$.

If $K > 0$, then the derivative $g_p^{'}(c)\,0$ has a root which leads to the maximum of price function (6.19). Hence

$$\begin{aligned} A_1 &= \left(-K\frac{r+\lambda}{L}\right)^{\frac{1}{n}} \\ A_2 &= \left(\frac{K\gamma_p}{\gamma_p+n}\right)^{\frac{1}{n}} \end{aligned} \tag{6.20}$$

and the perpetual price when $x > A_2$ is

$$P(x) = \frac{\left(-A_2^n + K\right) A_2^{\gamma_p}}{x^{\gamma_p}}. \tag{6.21}$$

If $L > 0$ and $M = 1$, then tenth statement of Lemma 8 holds due to Lemma 6. Therefore limit (6.13) is infinitely large which means that the immediate exercise is never optimal, i.e. $A_2 = 0$. Hence, the perpetual price is $P(x) = \infty$.[2] If $K \le 0$, then formula (6.12) leads to $A_1 = 0$. Otherwise, if $K > 0$, then

$$A_1 = \left(K\frac{r+\lambda}{L}\right)^{\frac{1}{n}}. \tag{6.22}$$

[2] The same conclusion can be made if we consider the derivative as a call-style—formula (6.14) tends to infinity when $c \to \infty$, because $\gamma_c < n$ when $L > 0$ (Lemma 4).

6.4 Results

We formulate the derived results in the following theorem.

Theorem 3 *Let the payoff be $G(x) = Mx^n + K$, where $n > 0$ and $M \in \{-1, 1\}$. Let also the constant L be defined as in Eq. (6.1). The following statements hold.*

1. *If $\{L = 0,\ K \geq 0\}$, $\{L < 0,\ M = 1,\ K \geq 0\}$, or $\{L > 0,\ M = -1,\ K \geq 0\}$, then the derivative is combined put-call and the immediate exercise is optimal everywhere. The call boundaries are $B_1 = B_2 = 0$, and the put ones are $A_1 = A_2 = \infty$. The perpetual price is given by formula (6.9).*
2. *If $\{L = 0,\ K < 0\}$, $\{L < 0,\ M = -1,\ K \leq 0\}$, or $\{L > 0,\ M = 1,\ K \leq 0\}$, then the derivative is again put-call style but the immediate exercise is never optimal. The call boundaries are $B_1 = B_2 = \infty$ and the put ones are $A_1 = A_2 = 0$. The perpetual price is presented by Eq. (6.11) in the first case, it is $P(x) = 0$ in the second one, and $P(x) = \infty$ in the third case.*
3. *If $\{L < 0,\ M = 1,\ K < 0\}$ or $\{L > 0,\ M = -1,\ K < 0\}$, then the derivative is call-style. The values of the optimal boundaries are given by formulas (6.15) and (6.17), respectively. The perpetual price is given by Eq. (6.9) if $x \geq B_2$ and by Eq. (6.16) or (6.18), otherwise (first and second case, respectively).*
4. *If $\{L < 0,\ M = -1,\ K > 0\}$ or $\{L > 0,\ M = 1,\ K > 0\}$, then we have a put-derivative. The boundaries are given in formulas (6.20) for the first case; the perpetual price is (6.9) if $x \leq A_2$ and (6.21) otherwise. The initial optimal boundary A_1 for the second case is given by formula (6.22) whereas the perpetual one is zero, $A_2 = 0$. The perpetual price in this case is $P(x) = \infty$.*

Remark 3 Let us mention that the price function has a left discontinuity w.r.t. the variable L in the point $L = 0$. This can be explained by the vanishing expectation in formula (4.5).

6.5 Finite maturity

We shall apply now the algorithm presented in Sect. 5. Suppose that the asset starts from a value $S_0 = x$. Therefore the strategy ζ, remind that it is the first hitting moment to an exponent of a piecewise linear function, can be viewed as the Brownian motion's first hitting to the level

$$\frac{1}{\sigma}\left(\left(a_i - r + \frac{\sigma^2}{2}\right)t + b_i - \log(x)\right) = C_i t_i + D_i$$

for

$$C_i = \frac{1}{\sigma}\left(a_i - r + \frac{\sigma^2}{2}\right)$$
$$D_i = \frac{b_i - \log(x)}{\sigma}.$$

We can derive payment (5.1) as

$$P(x; \{t_0, \ldots, t_l\}; \{c_0, \ldots, c_l\}) = E^x\left[e^{-(r+\lambda)(\zeta \wedge T)} G\left(S_{\zeta \wedge T}\right)\right]$$
$$= E^x\left[e^{-(r+\lambda)\zeta}\left(MS_\zeta^n + K\right) I_{\zeta < T}\right] + E^x\left[e^{-(r+\lambda)T}(MS_T^n + K) I_{\zeta \geq T}\right]$$

$$= KE\left[e^{-\alpha_1\zeta} I_{\zeta<T}\right] + Mx^n \sum_{m=1}^{l} e^{n\sigma D_m} E\left[e^{-\alpha_{2,m}\zeta} I_{t_{m-1}<\zeta\leq t_m}\right]$$
$$+Ke^{-\alpha_1 T} P(\zeta \geq T) + Mx^n e^{-\alpha_3 T} E\left[e^{n\sigma B_T} I_{\zeta\geq T}\right] \tag{6.23}$$

for

$$\alpha_1 = r + \lambda$$
$$\alpha_{2,m} = (r+\lambda) - n\left(r - \frac{\sigma^2}{2}\right) - nC_m\sigma = n\frac{\sigma^2}{2} - (n-1)r - nC_m\sigma + \lambda$$
$$\alpha_3 = \lambda + n\frac{\sigma^2}{2} - (n-1)r.$$

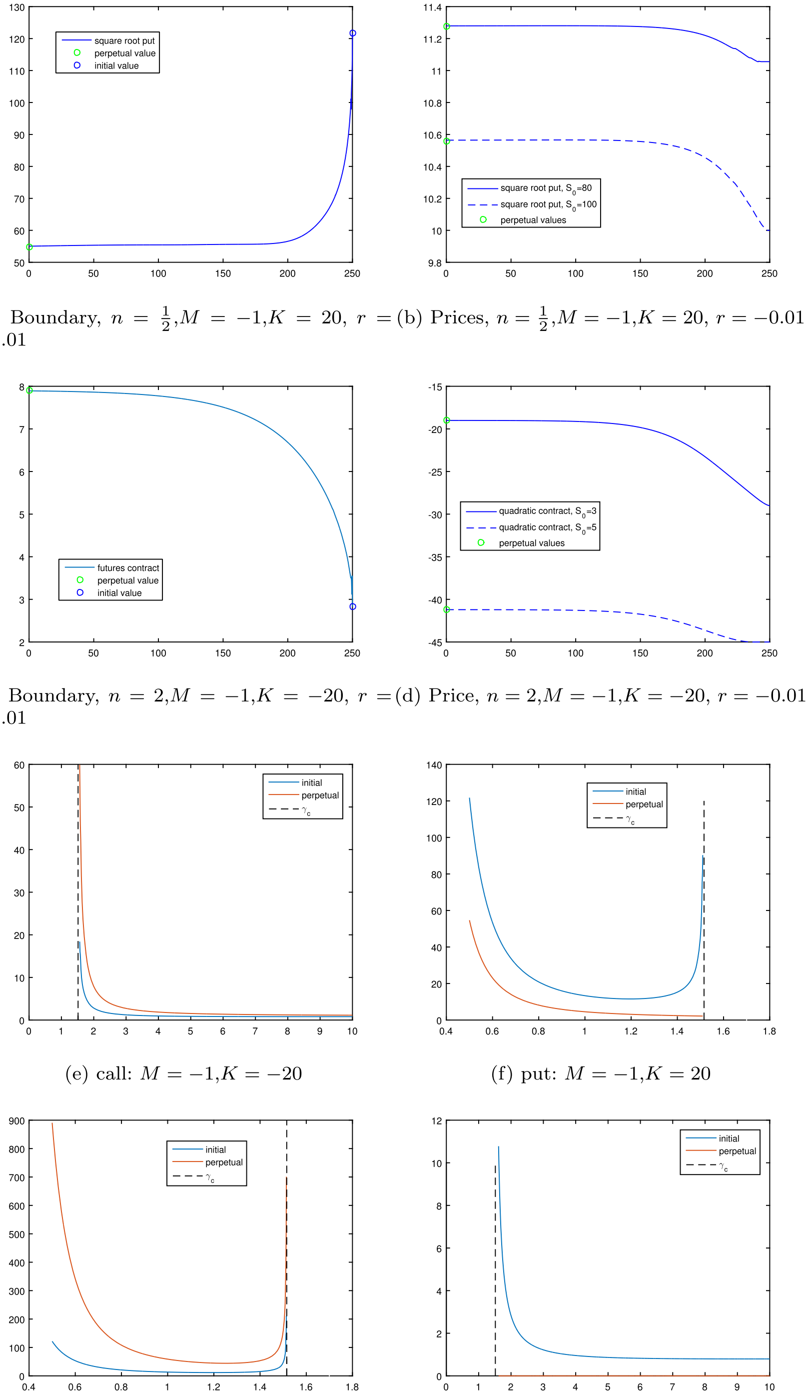
If we have a put-style derivative, then hitting is below and thus expectations in formula (6.23) can be derived through Propositions 13 and 14. If the derivative is call-style, then we have a hitting above problem. The corresponding expectations can be derived by symmetrical arguments—see Zaevski (2020b).

6.6 Numerical experiments

Suppose that the risk-free rate is $r = -0.01$, the additional discount rate is $\lambda = 0.03$, and the volatility is $\sigma = 0.3$. We investigate first the square root, i.e. $n = 0.5$. Also we consider $M = -1$ and $K = 20$. These values lead to $L = -0.0362$—the parameter L is given in formula (6.1). This means that the derivative is put-style since $ML > 0$. We present the optimal boundary at Fig. 1a. The initial and perpetual values derived via formulas (6.20) are $c(0) = 121.7598$ and $c(\infty) = 54.6700$. The prices are presented at Fig. 1b—the solid line is for $S_0 = \$80$ and the dashed one is for $S_0 = \$100$. The perpetual prices are \$11.2743 and \$10.5602, respectively.

Suppose now that $M = -1$, $K = -20$, and $n = 2$. Using formula (6.1) we obtain $L = 0.0500$ and therefore we have a call-style derivative. The optimal boundary is presented in Fig. 1c. The initial value is $c(0) = 2.8284$ and the perpetual one is $c(\infty) = 7.9093$. The prices for initial asset values $S_0 = \$3$ and $S_0 = \$5$ can be seen in Fig. 1d. The corresponding perpetual prices are negative, $-\$18.9980$ and $-\$41.2019$, respectively.

Next we investigate the behavior of the derivatives w.r.t. the power n. Let us consider the parameter L given in Eq. (6.1) as a function of n. Lemma 4 shows that $L(n) < 0$ for $n < \gamma_c$, $L(\gamma_c) = 0$, and $L(n) > 0$ for $n > \gamma_c$—the constant γ_c is given in formulas (4.4). In Table 1, we summarize the results presented in Theorem 3. We give only one pair of boundaries when the derivative exhibits a combined feature. We present the initial (blue one) and perpetual (red one) boundaries w.r.t. the power n in the following figures—Fig. 1e for $\{M = -1,\ K = -20\}$, Fig. 1f for $\{M = -1,\ K = 20\}$, Fig. 1g for $\{M = 1,\ K = -20\}$, and Fig. 1h for $\{M = 1,\ K = 20\}$. We conclude that we have a call-style derivative for lower values of n when $M = 1$ and $K < 0$ and for the larger ones when $M = -1$ and $K < 0$. On the other hand, the derivative is put-style for lower n's if $M = -1$ and $K > 0$ and for the higher values if $M = 1$ and $K > 0$. Also, for the call derivatives we have a continuity of the boundaries including in the point $n = \gamma_c$ (the value in this point is the infinity). The same is true for the put-derivatives but only for the initial values. For the perpetual ones we have a discontinuity in the point $n = \gamma_c$—see also Remark 3. If $M = -1$ and $K = 20$, then $\lim_{n\to\gamma_{c-}} c(\infty) = 2.1733$, but $c(\infty) = \infty$ for $n \geq \gamma_c$. On the other hand if $M = 1$ and $K = 20$, then $\lim_{n\to\gamma_{c+}} c(\infty) = 0$, but $c(\infty) = \infty$ for $n \leq \gamma_c$.



(a) Boundary, $n = \frac{1}{2}, M = -1, K = 20, r = -0.01$

(b) Prices, $n = \frac{1}{2}, M = -1, K = 20, r = -0.01$

(c) Boundary, $n = 2, M = -1, K = -20, r = -0.01$

(d) Price, $n = 2, M = -1, K = -20, r = -0.01$

(e) call: $M = -1, K = -20$

(f) put: $M = -1, K = 20$

(g) call: $M = 1, K = -20$

(h) put: $M = 1, K = 20$

Fig. 1 Put optimal boundaries and prices

Table 1 Derivative's characterization w.r.t. the power n

	Style	Initial boundary	Perpetual boundary	Perp. price $S_0 < c(\infty)$	Perp. price $S_0 \geq c(\infty)$
Parameters $M = -1$, $K < 0$					
$n \leq \gamma_c$	Put-call	$B_1 \equiv c(0) = \infty$	$B_2 \equiv c(\infty) = \infty$	0	–
$n > \gamma_c$	Call	(6.17)	(6.17)	(6.18)	(6.9)
Parameters $M = -1$, $K > 0$					
$n < \gamma_c$	Put	(6.20)	(6.20)	(6.9)	(6.21)
$n \geq \gamma_c$	Put-call	$A_1 \equiv c(0) = \infty$	$A_2 \equiv c(\infty) = \infty$	(6.9)	–
Parameters $M = 1$, $K < 0$					
$n < \gamma_c$	Call	(6.15)	(6.15)	(6.16)	(6.9)
$n \geq \gamma_c$	Put-call	$B_1 \equiv c(0) = \infty$	$B_2 \equiv c(\infty) = \infty$	∞	–
Parameters $M = 1$, $K > 0$					
$n \leq \gamma_c$	Put-call	$A_1 \equiv c(0) = \infty$	$A_2 \equiv c(\infty) = \infty$	(6.9)	–
$n > \gamma_c$	Put	(6.22)	$A_2 \equiv c(\infty) = 0$	∞	–

7 American call options

We discuss in this section the American call options in the light of the presented before general scheme. The main problem is that the payoff function $G(x) = (x - K)^+$ is not differentiable at the strike. To overcome this problem we approximate it by the twice differentiable functions $G_\epsilon(x)$ defined as

$$G_\epsilon(x) = \begin{cases} 0, & \text{if } x < K \\ \frac{(x-K)^2}{2\epsilon}, & \text{if } K \leq x < K + \epsilon \\ x - K - \frac{\epsilon}{2}, & \text{if } K + \epsilon \leq x. \end{cases} \tag{7.1}$$

These approximations can be seen at Fig. 2a—the strike is assumed to be $K = 20$ and the values of ϵ are $\epsilon \in \{0.1, 0.5, 1\}$. Thus the operator $\mathcal{B}$ applied to the function $G_\epsilon(\cdot)$ can be written as

$$(\mathcal{B}G_\epsilon)(x) = \begin{cases} 0, & \text{if } x < K \\ \frac{1}{2\epsilon}\left[\left(r - \lambda + \sigma^2\right)x^2 + 2\lambda K x - (r + \lambda) K^2\right], & \text{if } K \leq x < K + \epsilon \\ -\lambda x + (r + \lambda)\left(K + \frac{\epsilon}{2}\right), & \text{if } K + \epsilon \leq x. \end{cases}$$

We can easily check that $(\mathcal{B}G_\epsilon)(K) = \frac{K^2\sigma^2}{2\epsilon}$ and therefore $(\mathcal{B}G_\epsilon)(x) > 0$ for $K \leq x < K + \epsilon$ when ϵ is small enough. Hence, if $(\mathcal{B}G_\epsilon)(x) < 0$, then $x \geq K + \epsilon$. Having in mind the form of the function $(\mathcal{B}G_\epsilon)(x)$ when $x \geq K + \epsilon$, we see that the second condition of Definition 1 is satisfied, and therefore the payoff (7.1) leads to a call-style derivative. Something more, we conclude that the optimal boundary at the maturity is

$$B_{1,\epsilon} = \begin{cases} K + \epsilon, & \text{if } r < \lambda \frac{\epsilon}{2K+\epsilon} \\ \frac{r+\lambda}{\lambda}\left(K + \frac{\epsilon}{2}\right), & \text{if } r \geq \lambda \frac{\epsilon}{2K+\epsilon}. \end{cases}$$

Note that if $\lambda = 0$, then $B_1 = \infty$. The way we derive the optimal boundary at the maturity is shown at Fig. 2b, c. The parameters are $\lambda = 0.03$, $K = 20$, $\epsilon = 1$, $\sigma = 0.3$, and $r = \pm 0.01$—the risk-free values are for the first and second case, respectively. The optimal boundary is $B_{1,\epsilon} = 27.3333$ when $r = 0.01$ and $B_{1,\epsilon} = 21$ when $r = -0.01$ and it is marked by a red circle.

Let us turn to the perpetual value B_2. We shall prove first that the expectation in formula (4.5) tends to zero. We can rewrite it as

$$E^x\left[e^{-(r+\lambda)T} G_\epsilon\left(S_T\right) I_{T<\zeta^c}\right] = e^{-(r+\lambda)T} E^x\left[\left(S_T - K - \frac{\epsilon}{2}\right) I_{T<\zeta^c, B_T > \frac{1}{\sigma}\ln\frac{K+\epsilon}{x} - \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)T}\right]$$
$$+ \frac{e^{-(r+\lambda)T}}{2\epsilon} E^x\left[\left(S_T - K\right)^2 I_{T<\zeta^c, B_T \in \left(\frac{1}{\sigma}\ln\frac{K}{x} - \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)T, \frac{1}{\sigma}\ln\frac{K+\epsilon}{x} - \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)T\right]}\right].$$

The second term above is zero since $r + \lambda > 0$ and $\left(S_T - K\right)^2 < \epsilon^2$. On the other hand, the first term can be divided into two parts—the first one for $\left(K + \frac{\epsilon}{2}\right)$ and the second one for S_T. The first term is zero and the second one is

$$\begin{aligned} &e^{-(r+\lambda)T} E^x\left[S_T I_{T<\zeta^c, B_T > \frac{1}{\sigma}\ln\frac{K+\epsilon}{x} - \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)T}\right] \\ &= x e^{-\left(\lambda + \frac{\sigma^2}{2}\right)T} E^x\left[e^{\sigma B_T} I_{T<\zeta^c, B_T > \frac{1}{\sigma}\ln\frac{K+\epsilon}{x} - \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)T}\right]. \end{aligned} \tag{7.2}$$

Having in mind Eq. (A.2) we see that the expectation above tends to infinity as $e^{\frac{\sigma^2}{2}T}$. Hence if $\lambda > 0$, then expectation (7.2) tends to zero. It lefts to consider the case $\lambda = 0$ and therefore $r > 0$ since $r + \lambda > 0$. Looking again at Eq. (A.2), we see that the coefficient before T in the normal distribution CDF is $d_1 - \sigma = -\left(\frac{r}{\sigma} + \frac{\sigma}{2}\right) < 0$, where d_1 is given by formulas (4.2). Hence expectation (7.2) again tends to zero.

We conclude that the derivative price under the assumption that the holder exercises when the asset reaches the values c, given by formula (4.5), turns to

$$P^c\left(x; c\right) = x^{\gamma_c} \frac{G_\epsilon\left(c\right)}{c^{\gamma_c}}. \tag{7.3}$$

The constant γ_c is given by formula (4.4). We need to consider the behavior of the function $f\left(c\right) = \frac{G_\epsilon(c)}{c^{\gamma_c}}$. Its maximum will lead to the optimal boundary $B_{2,\epsilon}$. Obviously $f\left(c\right) = 0$ when $c < K$. Suppose that $K \le c < K + \epsilon$. We have for the derivative of the function $f\left(\cdot\right)$

$$f'\left(c\right) = \left(c - K\right) \frac{2c - \gamma_c\left(c - K\right)}{2\epsilon c^{\gamma_c + 1}}.$$

Therefore $f'\left(c\right) > 0$ in the whole interval $[K, K + \epsilon)$ for small enough ϵ's and thus the function $f\left(c\right)$ increases in this interval. It lefts to consider the case $K + \epsilon \le c$. The derivative $f'\left(\cdot\right)$ turns to

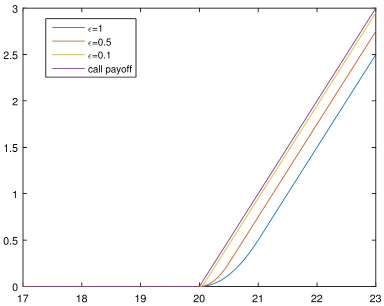
$$f'\left(c\right) = \frac{\gamma_c\left(K + \frac{\epsilon}{2}\right) - \left(\gamma_c - 1\right)c}{c^{\gamma_c + 1}}.$$

Hence, the function $f\left(c\right)$ achieves its maximum for small enough values of ϵ at the point

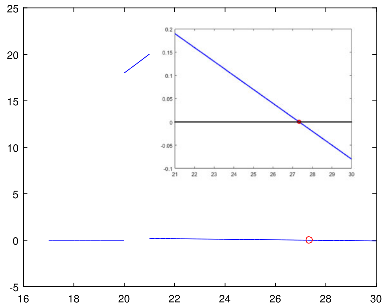
$$B_{2,\epsilon} = \frac{\gamma_c}{\gamma_c - 1}\left(K + \frac{\epsilon}{2}\right). \tag{7.4}$$

The way we derive the perpetual optimal boundary can be seen at Fig. 2d. The parameters are as above, we choose $r = 0.01$ for the risk-free rate. The optimal boundary is presented by a red point and it is $B_{2,\epsilon} = 70.6525$. Estimating value (7.4) in formula (7.3) and having in mind that $B_{2,\epsilon} > K + \frac{\epsilon}{2}$ we derive for the price when $x < B_{2,\epsilon}$

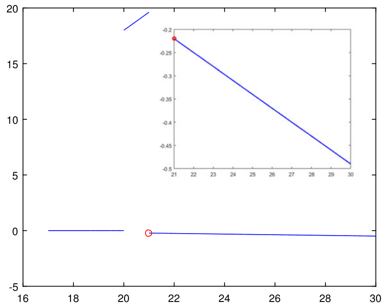
$$P^c\left(x; B_{2,\epsilon}\right) = \left(\frac{x}{\gamma_c}\right)^{\gamma_c}\left(\frac{\gamma_c - 1}{K + \frac{\epsilon}{2}}\right)^{\gamma_c - 1}.$$



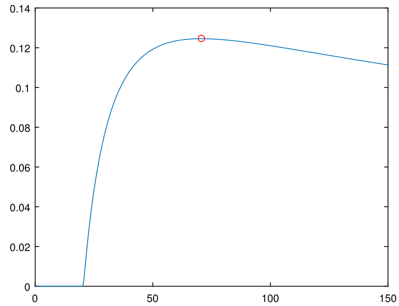
(a) Approximations of the call payoff



(b) Optimal boundary at the maturity, $r \geq \lambda \frac{\epsilon}{2K+\epsilon}$.



(c) Optimal boundary at the maturity, $r < \lambda \frac{\epsilon}{2K+\epsilon}$.



(d) Perpetual optimal boundary

Fig. 2 Call options

If we take the limit $\epsilon \to 0$, we derive the endpoints of the optimal boundaries as

$$B_1 = \begin{cases} K, & \text{if } r < 0 \\ \frac{r+\lambda}{\lambda} K, & \text{if } r \geq 0. \end{cases}$$
$$B_2 = \frac{\gamma_c}{\gamma_c - 1} K.$$

We can approximate now the whole optimal boundary—see Zaevski (2021). Having in mind Proposition 1, considering λ as a dividend rate, and changing the risk-free rate to $\overline{r} = r + \lambda$ we derive the well-known formulas for the optimal boundary endpoints—see for example Kim (1990).

We may examine the American put options in a similar manner approximating the put payoff $(K - x)^+$ by the functions

$$G_\epsilon(x) = \begin{cases} K + \frac{\epsilon}{2} - x, & \text{if } x < K - \epsilon \\ \frac{(K-x)^2}{2\epsilon}, & \text{if } K - \epsilon \leq x < K \\ 0, & \text{if } K \leq x. \end{cases}$$

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Data availability No data is used during the work on this article.

Declarations

Conflict of interest The author declares no competing interests.

Appendix A: Some hitting time properties

We present below some propositions for the Brownian motion's first hitting to a continuous piecewise linear function $c(t) = (a_m t + b_m) I_{t_{m-1}<t\leq t_m}$ w.r.t. the time grid $0 \equiv t_0 < t_1 < \cdots < t_n \equiv T$. We denote the values at the nodes by $c_m = a_m t_m + b_m$. We consider the lower hitting problem, i.e. $c(0) < 0$. The higher one is analogous due to the symmetry of the Brownian motion. We shall use the notation $c(t) = at + b$ if the boundary is linear.

Proposition 10 *Let $c(t) = at + b$ be a linear function. If $b = c(0) < 0$ and $z > c(T)$, then the Laplace transform of the first hitting moment ζ of a Brownian motion to the function $c(t)$ is*

$$\begin{aligned} V(\alpha, z, T) &\equiv E\left[e^{\alpha B_T} I_{B_T<z, T<\zeta}\right] \\ &= \exp\left(\frac{T\alpha^2}{2}\right)\left[\begin{array}{l} N\left(\frac{z-T\alpha}{\sqrt{T}}\right) - N\left(\frac{c(T)-T\alpha}{\sqrt{T}}\right) \\ -e^{2b(\alpha-a)}\left(N\left(\frac{z-T\alpha-2b}{\sqrt{T}}\right) - N\left(\frac{c(T)-T\alpha-2b}{\sqrt{T}}\right)\right) \end{array}\right]. \end{aligned} \tag{A.1}$$

Otherwise, if $b = c(0) > 0$ and $z < c(T)$, then

$$\begin{aligned} V(\alpha, z, T) &\equiv E\left[e^{\alpha B_T} I_{B_T>z, T<\zeta}\right] \\ &= \exp\left(\frac{T\alpha^2}{2}\right)\left[\begin{array}{l} N\left(\frac{c(T)-T\alpha}{\sqrt{T}}\right) - N\left(\frac{z-T\alpha}{\sqrt{T}}\right) \\ +e^{2b(\alpha-a)}\left(N\left(\frac{z-T\alpha-2b}{\sqrt{T}}\right) - N\left(\frac{c(T)-T\alpha-2b}{\sqrt{T}}\right)\right) \end{array}\right]. \end{aligned} \tag{A.2}$$

Proof The proof is a consequence of theorem 3.2 from Zaevski (2020b) and from the fact that the Brownian motion is a symmetric process. □

Proposition 11 *The probability of $\zeta < T$ is given by the equation*

$$g(T; a, b) \equiv P(\zeta < T) = N\left(\frac{aT+b}{\sqrt{T}}\right) + \exp(-2ab)\, N\left(\frac{-aT+b}{\sqrt{T}}\right). \tag{A.3}$$

Proof The proof can be found in Wang and Pötzelberger (1997, equation (3)), or in Zaevski (2020b, proposition 3.1). □

Proposition 12 *Let $\alpha > 0$. The truncated Laplace transform of ζ is*

$$L(T, \alpha; a, b) = E\left[e^{-\alpha\tau} I_{\zeta<T}\right] = e^{-b\left(\sqrt{a^2+2\alpha}+a\right)} g\left(T; -\sqrt{a^2+2\alpha}, b\right), \tag{A.4}$$

where the function $g(\cdot)$ is given in Eq. (A.3).

Proof See Zaevski (2020b, theorem 3.1). □

Proposition 13 *If the boundary is piecewise linear and* $\alpha > 0$*, then the truncated Laplace transform in the interval* (t_{m-1}, t_m) *is*

$$E\left[e^{-\alpha\tau} I_{\tau\in(t_{m-1},t_m)}\right] = \int_{c_1,\dots,c_{m-1}}^{\infty} \begin{pmatrix} \prod_{i=1}^{m-1}\left(1-\exp\left(-\frac{2(c_{i-1}-x_{i-1})(c_i-x_i)}{t_i-t_{i-1}}\right)\right) \\ \prod_{i=1}^{m-1}\frac{\left(\exp\left(-\frac{(x_i-x_{i-1})^2}{2(t_i-t_{i-1})}\right)\right)}{\sqrt{2\pi(t_i-t_{i-1})}} \\ e^{-\alpha t_{m-1}} L\left(t_m-t_{m-1},\alpha;a_m,c_{m-1}-x_{m-1}\right) \end{pmatrix} dx_1\dots dx_{m-1},$$

where the function $L(\cdot)$ *is given in Eq.* (A.4)*.*

Proof The proof can be found in Zaevski (2020b, theorem 4.1). □

Corollary 2 *If* $z > c(T)$*, then*

$$U(z,T;c(\cdot)) \equiv P\left(B_T < z, I_{T<\zeta}\right) = N\left(\frac{z}{\sqrt{T}}\right) - N\left(\frac{c(T)}{\sqrt{T}}\right) - e^{-2ba}\left[N\left(\frac{z-2b}{\sqrt{T}}\right) - N\left(\frac{c(T)-2b}{\sqrt{T}}\right)\right]. \tag{A.5}$$

Proof This statement is just formula (A.1) for $\alpha = 0$. □

Proposition 14 *The corresponding to* (A.1) *and* (A.5) *formulas when the boundary is piecewise linear are*

$$E\left[e^{\alpha B_T} I_{B_T<z,\Phi_T=1}\right] = \int_{c_1,\dots,c_{n-1}}^{\infty} \begin{pmatrix} \prod_{i=1}^{n-1}\left(1-\exp\left(-\frac{2(c_{i-1}-x_{i-1})(c_i-x_i)}{t_i-t_{i-1}}\right)\right) \\ \prod_{i=1}^{n-1}\frac{\left(\exp\left(-\frac{(x_i-x_{i-1})^2}{2(t_i-t_{i-1})}\right)\right)}{\sqrt{2\pi(t_i-t_{i-1})}} \\ e^{\alpha x_{n-1}} V\left(\alpha, z-x_{n-1}, t_n-t_{n-1}; c_{n-1}(\cdot)-x_{n-1}\right) \end{pmatrix} dx_1\dots dx_{n-1}$$

and

$$P\left(B_T < z, \Phi_T = 1\right) = \int_{c_1,\dots,c_{n-1}}^{\infty} \begin{pmatrix} \prod_{i=1}^{n-1}\left(1-\exp\left(-\frac{2(c_{i-1}-x_{i-1})(c_i-x_i)}{t_i-t_{i-1}}\right)\right) \\ \prod_{i=1}^{n-1}\frac{\left(\exp\left(-\frac{(x_i-x_{i-1})^2}{2(t_i-t_{i-1})}\right)\right)}{\sqrt{2\pi(t_i-t_{i-1})}} \\ e^{\alpha x_{n-1}} U\left(z-x_{n-1}, t_n-t_{n-1}; c(\cdot)-x_{n-1}\right) \end{pmatrix} dx_1\dots dx_{n-1},$$

where the functions $V(\cdot)$ *and* $U(\cdot)$ *are given by Eqs.* (A.1) *and* (A.5)*.*

Proof For the first statement see Zaevski (2020b, theorem 4.2). The second one is obtained for $\alpha = 0$. □

Appendix B: Some limits

We shall prove first several lemmas. Let us denote again by ζ^c the first hitting moment of the underlying asset to the value c and the function $d(\cdot)$ be defined in Eqs. (4.1) and (4.2). Let us denote by $N(\cdot)$ the cumulative distribution function of the standard normal distribution.

Lemma 7 *Suppose that $k > 0$ and $m < 0$ are constants. The following statements hold.*

1. *If $k > \frac{m^2}{2}$, then $\lim_{T\to\infty} e^{kT} N\left(m\sqrt{T}\right) = \infty$.*
2. *If $k \leq \frac{m^2}{2}$, then $\lim_{T\to\infty} e^{kT} N\left(m\sqrt{T}\right) = 0$.*

Proof The proof follows a result for the Mills' ratio for $x > 0$

$$\frac{x}{x^2+1} \leq e^{\frac{x^2}{2}} \int_x^\infty e^{-\frac{u^2}{2}} du \leq \frac{1}{x},$$

which is provided in Gordon (1941). □

Lemma 8 *Let θ be a positive number.*

1. *If $\left\{ k < -\frac{\theta^2}{2} \right\}$, then $\lim_{T\to\infty} e^{kT} E^x \left[e^{\theta B_T} I_{T<\zeta^c}\right] = 0$.*
2. *If $\left\{d_1{=}\theta, k{=}-\frac{\theta^2}{2}\right\}$, then $\lim_{T\to\infty} e^{kT} E^x \left[e^{\theta B_T} I_{T<\zeta^c}\right] = 0$ and if $\left\{d_1{=}\theta, k > -\frac{\theta^2}{2}\right\}$, then $\lim_{T\to\infty} e^{kT} E^x \left[e^{\theta B_T} I_{T<\zeta^c}\right] = \infty$.*
3. *If $\left\{c > x,\ d_1 < \theta,\ k = -\frac{\theta^2}{2}\right\}$, then $\lim_{T\to\infty} e^{kT} E^x \left[e^{\theta B_T} I_{T<\zeta^c}\right] = 0$.*
4. *If $\left\{c > x,\ d_1 > \theta,\ k = -\frac{\theta^2}{2}\right\}$, then $\lim_{T\to\infty} e^{kT} E^x \left[e^{\theta B_T} I_{T<\zeta^c}\right] = 1 - e^{2d_2(\theta-d_1)}$.*
5. *If $\left\{c > x,\ d_1 > \theta,\ k > -\frac{\theta^2}{2}\right\}$, then $\lim_{T\to\infty} e^{kT} E^x \left[e^{\theta B_T} I_{T<\zeta^c}\right] = \infty$.*
6. *If $\left\{c > x,\ d_1 < \theta,\ -\frac{\theta^2}{2} < k \leq \frac{d_1^2}{2} - \theta d_1\right\}$, then $\lim_{T\to\infty} e^{kT} E^x \left[e^{\theta B_T} I_{T<\zeta^c}\right] = 0$.*
7. *If $\left\{c > x,\ d_1 < \theta,\ \frac{d_1^2}{2} - \theta d_1 < k\right\}$, then $\lim_{T\to\infty} e^{kT} E^x \left[e^{\theta B_T} I_{T<\zeta^c}\right] = \infty$.*
8. *If $\left\{c < x,\ d_1 > \theta,\ k = -\frac{\theta^2}{2}\right\}$, then $\lim_{T\to\infty} e^{kT} E^x \left[e^{\theta B_T} I_{T<\zeta^c}\right] = 0$.*
9. *If $\left\{c < x,\ d_1 < \theta,\ k = -\frac{\theta^2}{2}\right\}$, then $\lim_{T\to\infty} e^{kT} E^x \left[e^{\theta B_T} I_{T<\zeta^c}\right] = 1 - e^{2d_2(\theta-d_1)}$.*
10. *If $\left\{c < x,\ d_1 < \theta,\ k > -\frac{\theta^2}{2}\right\}$, then $\lim_{T\to\infty} e^{kT} E^x \left[e^{\theta B_T} I_{T<\zeta^c}\right] = \infty$.*
11. *If $\left\{c < x,\ d_1 > \theta,\ -\frac{\theta^2}{2} < k \leq \frac{d_1^2}{2} - \theta d_1\right\}$, then $\lim_{T\to\infty} e^{kT} E^x \left[e^{\theta B_T} I_{T<\zeta^c}\right] = 0$.*
12. *If $\left\{c < x,\ d_1 > \theta,\ \frac{d_1^2}{2} - \theta d_1 < k\right\}$, then $\lim_{T\to\infty} e^{kT} E^x \left[e^{\theta B_T} I_{T<\zeta^c}\right] = \infty$.*

Proof Suppose first that $c > x$. Using formula (A.2) for $z = -\infty$ we obtain

$$E^x\left[e^{\theta B_T} I_{T<\zeta^c}\right] = \exp\left(\frac{T\theta^2}{2}\right)\left[N\left(\frac{d(T) - T\theta}{\sqrt{T}}\right) - e^{2d_2(\theta-d_1)} N\left(\frac{d(T) - T\theta - 2d_2}{\sqrt{T}}\right)\right]. \tag{B.1}$$

The first five statements are immediate consequence of formula (B.1). The sixth and seventh points follow from Lemma 7 applied to formula (B.1).

Suppose now that $c < x$. Formula (A.1) written for $z = \infty$ leads to

$$\begin{aligned} &E^x\left[e^{\theta B_T} I_{T<\zeta^c}\right] \\ &\quad = \exp\left(\frac{T\theta^2}{2}\right)\left[N\left(-\frac{d(T) - T\theta}{\sqrt{T}}\right) - e^{2d_2(\theta-d_1)} N\left(-\frac{d(T) - T\theta - 2d_2}{\sqrt{T}}\right)\right]. \end{aligned}$$

The rest of the proof is identical to the previous case. □

Appendix C: Finite difference terms

$$\begin{aligned} F_t &= \frac{F(m-1,n) - F(m,n)}{\Delta t} \\ F &= \frac{F(m-1,n) + F(m,n)}{2} \\ F_x &= \frac{F(m-1,n) - F(m-1,n-1) + F(m,n) - F(m,n-1)}{2\Delta x} \\ F_{xx} &= \frac{F(m-1,n+1) - 2F(m-1,n) + F(m-1,n-1)}{2(\Delta x)^2} \\ &\quad + \frac{F(m,n+1) - 2F(m,n) + F(m,n-1)}{2(\Delta x)^2}. \end{aligned} \tag{C.1}$$

$$\begin{aligned} 0 &= \frac{F(m-1,n) - F(m,n)}{\Delta t} \\ &\quad + \frac{1}{2} r x_n \frac{F(m-1,n) - F(m-1,n-1) + F(m,n) - F(m,n-1)}{\Delta x} \\ &\quad + \frac{1}{4}\sigma^2 x_n^2 \left(\begin{array}{l} \frac{F(m-1,n+1)-2F(m-1,n)+F(m-1,n-1)}{(\Delta x)^2} \\ + \frac{F(m,n+1)-2F(m,n)+F(m,n-1)}{(\Delta x)^2} \end{array}\right) \\ &\quad - \frac{1}{2} r\left(F(m-1,n) + F(m,n)\right). \end{aligned} \tag{C.2}$$

- If $n = 2$, then

$$\begin{aligned} &F(m,n)\left(\frac{1}{\Delta t} - \frac{1}{2}\frac{r x_n}{\Delta x} + \frac{1}{2}\frac{\sigma^2 x_n^2}{(\Delta x)^2} + \frac{1}{2} r\right) \\ &\quad - F(m,n+1)\frac{1}{4}\frac{\sigma^2 x_n^2}{(\Delta x)^2} \\ &\quad = F(m-1,n-1)\left(-\frac{1}{2}\frac{r x_n}{\Delta x} + \frac{1}{4}\frac{\sigma^2 x_n^2}{(\Delta x)^2}\right) \end{aligned}$$

$$+F(m-1,n)\left(\frac{1}{\Delta t}+\frac{1}{2}\frac{rx_n}{\Delta x}-\frac{1}{2}\frac{\sigma^2 x_n^2}{(\Delta x)^2}-\frac{1}{2}r\right)$$
$$+F(m-1,n+1)\frac{1}{4}\frac{\sigma^2 x_n^2}{(\Delta x)^2}$$
$$-F(m,1)\left(\frac{1}{2}\frac{rx_n}{\Delta x}-\frac{1}{4}\frac{\sigma^2 x_n^2}{(\Delta x)^2}\right). \tag{C.3}$$

- If $2 < n < k_m - 1$, then

$$\begin{aligned}
&F(m,n-1)\left(\frac{1}{2}\frac{rx_n}{\Delta x}-\frac{1}{4}\frac{\sigma^2 x_n^2}{(\Delta x)^2}\right)\\
&\quad+F(m,n)\left(\frac{1}{\Delta t}-\frac{1}{2}\frac{rx_n}{\Delta x}+\frac{1}{2}\frac{\sigma^2 x_n^2}{(\Delta x)^2}+\frac{1}{2}r\right)\\
&\quad-F(m,n+1)\frac{1}{4}\frac{\sigma^2 x_n^2}{(\Delta x)^2}\\
&=F(m-1,n-1)\left(-\frac{1}{2}\frac{rx_n}{\Delta x}+\frac{1}{4}\frac{\sigma^2 x_n^2}{(\Delta x)\,2}\right)\\
&\quad+F(m-1,n)\left(\frac{1}{\Delta t}+\frac{1}{2}\frac{rx_n}{\Delta x}-\frac{1}{2}\frac{\sigma^2 x_n^2}{(\Delta x)^2}-\frac{1}{2}r\right)\\
&\quad+F(m-1,n+1)\frac{1}{4}\frac{\sigma^2 x_n^2}{(\Delta x)^2}.
\end{aligned} \tag{C.4}$$

- If $n = k_m - 1$, then

$$\begin{aligned}
&F(m,n-1)\left(\frac{1}{2}\frac{rx_n}{\Delta x}-\frac{1}{4}\frac{\sigma^2 x_n^2}{(\Delta x)^2}\right)\\
&\quad+F(m,n)\left(\frac{1}{\Delta t}-\frac{1}{2}\frac{rx_n}{\Delta x}+\frac{1}{2}\frac{\sigma^2 x_n^2}{(\Delta x)^2}+\frac{1}{2}r\right)\\
&=F(m-1,n-1)\left(-\frac{1}{2}\frac{rx_n}{\Delta x}+\frac{1}{4}\frac{\sigma^2 x_n^2}{(\Delta x)^2}\right)\\
&\quad+F(m-1,n)\left(\frac{1}{\Delta t}+\frac{1}{2}\frac{rx_n}{\Delta x}-\frac{1}{2}\frac{\sigma^2 x_n^2}{(\Delta x)^2}-\frac{1}{2}r\right)\\
&\quad+F(m-1,n+1)\frac{1}{4}\frac{\sigma^2 x_n^2}{(\Delta x)^2}\\
&\quad+F(m,n+1)\frac{1}{4}\frac{\sigma^2 x_n^2}{(\Delta x)^2}.
\end{aligned} \tag{C.5}$$

References

Abdi-Mazraeh S, Khani A (2018) An efficient computational algorithm for pricing European, barrier and American options. Comput Appl Math 37(4):4856–4876

Ayache E, Forsyth PA, Vetzal KR (2003) Valuation of convertible bonds with credit risk. J Deriv 11(1):9–29. ISSN 1074-1240. https://doi.org/10.3905/jod.2003.319208. http://jod.iijournals.com/content/11/1/9

Bather J (1970) Optimal stopping problems for Brownian motion. Adv Appl Probab 2(2):259–286

Battauz A, Rotondi F (2024) Optimal liquidation policies of redeemable shares. SSRN Electron J. https://doi.org/10.2139/ssrn.4681818

Battauz A, De Donno M, Sbuelz A (2022a) On the exercise of American quanto options. North Am J Econ Finance 62:101738. ISSN 1062-9408. https://doi.org/10.1016/j.najef.2022.101738. https://www.sciencedirect.com/science/article/pii/S1062940822000870

Battauz A, De Donno M, Gajda J, Sbuelz A (2022b) Optimal exercise of American put options near maturity: a new economic perspective. Rev Deriv Res 25:23–46. https://doi.org/10.1007/s11147-021-09180-w

Black F, Scholes M (1973) The pricing of options and corporate liabilities. J Polit Econ 81(3):637–659

Borodin AN, Salminen P (2015) Handbook of Brownian motion—facts and formulae. Probability and Its Applications. Birkhäuser, Basel, ISBN 9783764367053

Brennan MJ, Schwartz ES (1977) Convertible bonds: valuation and optimal strategies for call and conversion. J Finance 32(5):1699–1715

Burnecki K, Giuricich MN, Palmowski ZB (2019) Valuation of contingent convertible catastrophe bonds - the case for equity conversion. Insur Math Econ 88:238–254. ISSN 0167-6687. https://doi.org/10.1016/j.insmatheco.2019.07.006. http://www.sciencedirect.com/science/article/pii/S0167668718301598

Carr P, Jarrow R, Myneni R (1992) Alternative characterizations of American put options. Math Finance 2(2):87–106. ISSN 1467-9965. https://doi.org/10.1111/j.1467-9965.1992.tb00040.x. http://dx.doi.org/10.1111/j.1467-9965.1992.tb00040.x

Chernogorova TP, Koleva MN, Valkov RL (2018) A two-grid penalty method for American options. Comput Appl Math 37(3):2381–2398

De Spiegeleer J, Schoutens W, Jabre P (2011) The handbook of convertible bonds: pricing. Strategies and risk management. The Wiley Finance Series. Wiley. ISBN 9781119978060

Gordon RD (1941) Values of Mills' ratio of area to bounding ordinate and of the normal probability integral for large values of the argument. Ann Math Stat 12:364–366

Heidari SS, Azari H (2018) A front-fixing finite element method for pricing American options under regime-switching jump-diffusion models. Comput Appl Math 37(3):3691–3707

Jacka SD (1991) Optimal stopping and the American put. Math Finance 1(2):1–14. ISSN 1467-9965. https://doi.org/10.1111/j.1467-9965.1991.tb00007.x. http://dx.doi.org/10.1111/j.1467-9965.1991.tb00007.x

Jeon J, Oh J (2019) Valuation of American strangle option: variational inequality approach. Discrete Contin Dyn Syst B 24(2):755

Jeon J, Kim G (2022) Analytic valuation formula for American strangle option in the mean-reversion environment. Mathematics 10(15). ISSN 2227-7390. https://doi.org/10.3390/math10152688. https://www.mdpi.com/2227-7390/10/15/2688

Kim IJ (1990) The analytic valuation of American options. Rev Financ Stud 3(4):547–572. ISSN 08939454, 14657368. http://www.jstor.org/stable/2962115

Lee J-K (2020) A simple numerical method for pricing American power put options. Chaos Solitons Fractals 139:110254. ISSN 0960-0779. https://doi.org/10.1016/j.chaos.2020.110254. http://www.sciencedirect.com/science/article/pii/S0960077920306500

Miao DWC, Lin XCS, Yu SHT (2016) A note on the never-early-exercise region of American power exchange options. Oper Res Lett 44(1):129–135. ISSN 0167-6377. https://doi.org/10.1016/j.orl.2015.12.011. https://www.sciencedirect.com/science/article/pii/S0167637715001704

Milanov K, Kounchev O, Fabozzi F (2019) A complete model for pricing coco bonds. J Fixed Income 29(3):53–67

Moradipour M, Yousefi SA (2018) Using a meshless kernel-based method to solve the Black–Scholes variational inequality of American options. Comput Appl Math 37(1):627–639

Peskir G, Shiryaev A (2006) Optimal Stopping and Free-Boundary Problems. Lectures in Mathematics. ETH Zürich, Birkhäuser, Basel. ISBN 9783764373900

Pham H (1997) Optimal stopping, free boundary, and American option in a jump-diffusion model. Appl Math Optim 35(2):145–164. ISSN 1432-0606. https://doi.org/10.1007/BF02683325

Qiu S (2020) American strangle options. Appl Math Finance 27(3):228–263

Shiryaev AN (2009) Optimal stopping rules. Stochastic modelling and applied probability. Springer, Berlin. ISBN 9783540841814

Tsiveriotis K, Fernandes C (1998) Valuing convertible bonds with credit risk. J Fixed Income 8(2):95

Van Moerbeke P (1973) On optimal stopping and free boundary problems. Adv Appl Probab 5(1):33–35

Wang L, Pötzelberger K (1997) Boundary crossing probability for Brownian motion and general boundaries. J Appl Probab 34(1):54–65

Wong D (1996) Generalized optimal stopping problems and financial markets. Chapman & Hall/CRC Research Notes in Mathematics Series. Taylor & Francis. ISBN 9780582304000. https://books.google.bg/books?id=uQdW8tADrsAC

Zaevski TS (2020a) Discounted perpetual game call options. Chaos Solitons Fractals 131:109503. ISSN 0960-0779. https://doi.org/10.1016/j.chaos.2019.109503. http://www.sciencedirect.com/science/article/pii/S0960077919304552

Zaevski TS (2020b)Laplace transforms for the first hitting time of a Brownian motion. Comptes rendus de l'Académie bulgare des Sciences 73(7):934–941. ISSN 2367-6248 (print), 2603-4832 (online). https://doi.org/10.7546/CRABS.2020.07.05

Zaevski TS (2021) A new approach for pricing discounted American options. Commun Nonlinear Sci Numer Simul 97:105752. ISSN 1007-5704. https://doi.org/10.1016/j.cnsns.2021.105752. https://www.sciencedirect.com/science/article/pii/S1007570421000630

Zhang Q, Song H, Yang C, Wu F (2020) An efficient numerical method for the valuation of American multi-asset options. Comput Appl Math 39(3):1–12

Zhang Q, Song H, Hao Y (2022) Semi-implicit FEM for the valuation of American options under the Heston model. Comput Appl Math 41(2):73